ON CERTAIN PROPERTIES OF MOTION IN THE PRESENCE OF GYROSCOPIC FORCES

(O NEKOTORYKH SVOISTVAKH DVIZHENIIA POD DEISTVIEM GIROSKOPICHEKIKH SIL)

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1. We shall consider the motion of a particle subject to the action of the so-called gyroscopic force $\mathbf{F}$, i.e. a force which is perpendicular to the velocity $\mathbf{v}$ of the particle. Further, we shall assume that such a force can be represented by a vector product

$$\mathbf{F} = \mathbf{v} \times \mathbf{\Phi} \tag{1.1}$$

where the vector $\mathbf{\Phi}$ depends on the coordinates of the particle only.

As an example consider the motion of an electron in a magnetic field subject to the action of the Lorentz force

$$\mathbf{F} = \frac{e}{c}\mathbf{v} \times \mathbf{H}, \qquad \mathbf{\Phi} = \frac{e}{c}\mathbf{H} \tag{1.2}$$

where $\mathbf{H}$ is the intensity of the magnetic field, e is the charge of the electron and c is the velocity of light.

As a second example consider a particle attracted by a fixed axis with a force proportional to the distance of the particle from the axis, and study the motion of this particle with respect to a coordinate system which rotates uniformly with angular velocity ω with respect to the above fixed axis. The law of relative motion then reads:

$$m\mathbf{w} = -k\mathbf{r} + \mathbf{J}_e + \mathbf{J}_c, \qquad \mathbf{J}_e = m\mathbf{r}\omega^2, \qquad \mathbf{J}_c = 2m\mathbf{v} \times \boldsymbol{\omega} \tag{1.3}$$

Here $\mathbf{r}$ is the vector directed from the particle to the axis along the shortest distance, $\mathbf{J}_e$ is the transport (centrifugal) inertia force and $\mathbf{J}_c$ is the Coriolis inertia force. If we put $k = m\omega^2$, then we have

$$m\mathbf{w} = 2m\mathbf{v} \times \boldsymbol{\omega}, \qquad \mathbf{\Phi} = 2m\boldsymbol{\omega}$$

Consider first some general properties of the motion of a particle subject to the action of a gyroscopic force. Let the equation of motion be of the form

$$\frac{d}{dt}(m\mathbf{v}) = \mathbf{v} \times \boldsymbol{\Phi} \tag{1.4}$$

where the mass $m = m(t, v)$ is a variable quantity which depends on the time t and the velocity $\mathbf{v}$. Then $\mathbf{v} = v\boldsymbol{\tau}$, $v = ds/dt$, where $\boldsymbol{\tau}$ is the unit vector along the tangent. We easily find that

$$\frac{d}{dt}(m\mathbf{v}) = \frac{d}{dt}(mv\boldsymbol{\tau}) = \boldsymbol{\tau}\frac{d(mv)}{dt} + mv\frac{d\boldsymbol{\tau}}{dt}$$

Since $d\boldsymbol{\tau}/dt = (d\boldsymbol{\tau}/ds)(ds/dt) = \mathbf{n}v/\rho$, then finally we have

$$\boldsymbol{\tau}\frac{d(mv)}{dt} + \mathbf{n}\frac{mv^2}{\rho} = \mathbf{v} \times \boldsymbol{\Phi} \tag{1.5}$$

Multiplying scalarly both sides of this equality by the unit vector $\boldsymbol{\tau}$, we obtain

$$\frac{d}{dt}(mv) = 0$$

From the last equation follows the law of conservation of the scalar quantity, called the quantity of motion [momentum]:

$$q = mv = \text{const} \tag{1.6}$$

Conversely, from this conservation law it follows that, due to (1.5), the force is directed along the principal normal to the trajectory. In the particular case where the mass depends only on the velocity, it follows from (1.6) that the motion is uniform and hence the mass is constant.

Now let the particle be subject to a potential force $\mathbf{F}_0 = -\operatorname{grad} V_0$ as well as to the gyroscopic force. Then

$$\frac{d(m\mathbf{v})}{dt} = -\operatorname{grad} V_0 + \mathbf{v} \times \boldsymbol{\Phi} \tag{1.7}$$

Multiplying this equality scalarly by the vector $d\mathbf{r} = \mathbf{v}\,dt$, we obtain

$$d\mathbf{r} \cdot \frac{d(m\mathbf{v})}{dt} = \mathbf{v} \cdot d(m\mathbf{v}) = -\operatorname{grad} V_0 \cdot d\mathbf{r} = -dV_0$$

Introducing the notation

$$\int mv\,dv = \lambda \tag{1.8}$$

we find

$$\mathbf{v} \cdot d(m\mathbf{v}) = d(\mathbf{v} \cdot m\mathbf{v}) - m\mathbf{v} \cdot d\mathbf{v} = d(mv^2) - mv\,dv = d(mv^2) - d\lambda$$

Hence, $d(mv^2 + V_0 - \lambda) = 0$. Thus, we have the law of conservation

$$2T + V_0 - \lambda = h \qquad \left(T = \frac{1}{2} mv^2\right) \tag{1.9}$$

In the case of constant mass $\lambda = (1/2)mv^2 = T$, and equality (1.9) expresses the law of conservation of the mechanical energy. In relativistic mechanics we have

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}, \qquad \lambda = m_0 \int \frac{vdv}{\sqrt{1 - v^2/c^2}} = -m_0 c^2 \sqrt{1 - \frac{v_2}{c^2}} \tag{1.10}$$

and in this case (1.9) expresses the law of conservation of energy of the electron (cf., for example, [2, Chapt. 10]):

$$h = mv^2 + V_0 - \lambda = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} + V_0 = mc^2 + V_0 \tag{1.11}$$

2. Now let the particle be subject to the action of an arbitrarily given force $\mathbf{F}_0$ and a gyroscopic force $\mathbf{F} = \mathbf{v} \times \mathbf{\Phi}$, where $\mathbf{\Phi}$ is a constant vector. In this section we shall consider the mass of the particle to be constant.

On the basis of (1.3) we can assert that in the presence of a gyroscopic force the motion of the particle is identical to the motion of this same particle with respect to a coordinate system which rotates uniformly with angular velocity $\omega = (1/2\, m)\mathbf{\Phi}$ about the axis coinciding with the vector $\mathbf{\Phi}$, the particle being subject to the action of a force of attraction to the axis of rotation, equal to $-mr\omega^2$, instead of a gyroscopic force.

If the vector $\mathbf{\Phi}$ is so small that the square of its modulus is negligible, then this supplementary force can be neglected.

In particular, in the case of the motion of an electron in a magnetic field, we have, as was said before, $\mathbf{\Phi} = (e/c)\mathbf{H}$. The quantity $\boldsymbol{\omega} = (e/2\,cm)\mathbf{H}$ is called Larmor's angular velocity (cf., for example, [1, Sect. 15]).

As an illustration consider the motion of a particle in the Oxy-plane subject to the action of a central force $\mathbf{F}_0$ and a gyroscopic force $\mathbf{F}$. Assume that the Oz-axis coincides with the constant direction of the vector $\mathbf{\Phi}$.

Using the expressions for the projections of the velocity and acceleration of the particle on the radius vector and the perpendicular to the latter

$$v_r = \dot{r}, \qquad v_s = r\dot{\theta}, \qquad w_r = \ddot{r} - r\dot{\theta}^2, \qquad w_s = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$$

we obtain the projections $F_r = r\Phi\dot{\theta}$, $F_s = -\dot{r}\Phi$ of the force $\mathbf{F} = \mathbf{v}\times\Phi$ and the equations of motion

$$m(\ddot{r} - r\dot{\theta}^2) = F_0 + r\dot{\theta}\Phi = F_0 + 2rm\omega\dot{\theta}, \qquad \frac{m}{r}\frac{d}{dt}(r^2\dot{\theta}) = -\dot{r}\Phi = -2m\dot{r}\omega \tag{2.1}$$

From the second equation we find the generalized area integral

$$\frac{d}{dt}[r^2(\dot{\theta} + \omega)] = 0, \qquad r^2(\dot{\theta} + \omega) = C \tag{2.2}$$

Solving this equation for $\dot{\theta}$ and substituting the expression so obtained into the first equation of (2.1), we obtain the law of motion of a particle in the radial direction

$$m\ddot{r} = F_0 + \frac{mC^2}{r^3} - mr\omega^2 \tag{2.3}$$

This is the same as if the particle were subject to the action not only of a central force but also of a force of attraction to the center, proportional to the distance from the latter, and a force of repulsion from this center, inversely proportional to the cube of this distance.

If the central force is a function of this distance only, so that $F_0 = -\operatorname{grad} V_0$, then

$$m\ddot{r} = -\operatorname{grad} V, \qquad V = V_0 + \frac{m}{2}\left(\frac{C}{r} + r\omega\right)^2 \tag{2.4}$$

These results may be useful for the investigation of motion of an electron in a cylindrical magnetron, the cathode and the anode of which are coaxial cylinders of radii r_1 and r_2, respectively, and the intensity $\mathbf{H}$ of the magnetic field is directed along the common axis of the cylinders. If ϕ_1 and $\phi_2 (> \phi_1)$ are the potentials of the cathode and anode, respectively, then it is easy to see that the potential V_0 of the electric field is

$$V_0 = e\left[(\varphi_2 - \varphi_1)\frac{\ln(r/r_1)}{\ln(r_2/r_1)} + \varphi_1\right] \tag{2.5}$$

In order to investigate the problem of hitting the anode by the electrons radiating from the cathode, apply Equation (2.3), putting

$$F_0 = -\operatorname{grad} V_0 = -\frac{e(\varphi_2 - \varphi_1)}{\ln(r_2/r_1)}\frac{1}{r}, \qquad \omega = \frac{eH}{2cm} \tag{2.6}$$

Since from (2.4) follows the equality $(1/2)m\dot{r}^2 + V = \text{const}$, then it is easy to obtain from here the condition that the trajectories of electrons radiating from the cathode with radial velocity v_0 hit the anode, namely

$$^1/_2\, mv_0^2 + V(r_1) = V(r_2)$$

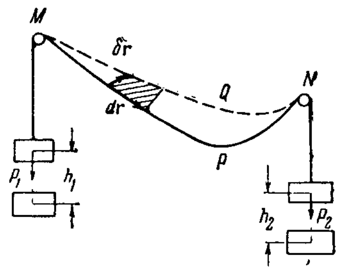
Here, by virtue of (2.2), we can write that $C = r_1^{\,2}\omega = r_1^{\,2} eH/2\,cm$.

3. Consider now a simple electro-mechanical analogy. Consider an ideal flexible inextensible string, each element ds of which is subject to the action of the force $\mathbf{F}_1 ds$. As it is well known, the equilibrium condition of the string is

$$\frac{d}{ds}(\mu\boldsymbol{\tau}) + \mathbf{F}_1 = 0 \qquad (3.1)$$

where the scalar quantity μ is the tension of the string. Let us compare this equation with that of the motion of the particle

$$\frac{d}{dt}(m\mathbf{v}) = \mathbf{F}$$



which can be rewritten in the form

$$\frac{d}{dt}(mv\boldsymbol{\tau}) = \frac{ds}{dt}\frac{d}{ds}(mv\boldsymbol{\tau}) = \mathbf{F}, \qquad \frac{d}{ds}(mv\boldsymbol{\tau}) = \frac{\mathbf{F}}{v}\,.$$

It is evident from this that the curve, which is the equilibrium form of the string, coincides with the trajectory of the particle moving under the action of the force $\mathbf{F} = -\mathbf{F}_1 v$. In particular, if the force acts along the normal to the string, the tension μ is constant in absolute value and equal to the constant quantity of motion of the particle

Let $\mathbf{F}_1 = -\boldsymbol{\tau}\times\boldsymbol{\Phi}$, and let the curve MPN be the equilibrium form of the string (Figure). At the points M and N the string hangs over infinitely small ideal pulleys, and at the free ends of the string are attached equal loads $P_1 = P_2 = \mu$. Let us impart to each element of the string a virtual displacement $\delta\mathbf{r}$, and let the string assume a new position MQN. Then the elementary work done by the force $\mathbf{F}_1 ds$ is

$$\mathbf{F}_1 ds\cdot\delta\mathbf{r} = -(\boldsymbol{\tau}\times\boldsymbol{\Phi}), \quad \delta\mathbf{r} ds = (d\mathbf{r}\times\delta\mathbf{r})\cdot\boldsymbol{\Phi}, \qquad d\mathbf{r} = \tau ds \qquad (3.2)$$

If we introduce the vector $d\mathbf{S} = d\mathbf{r}\times\delta\mathbf{r}$, then its modulus $dS = |d\mathbf{r}\times\delta\mathbf{r}|$ is equal to the area element dS, shaded in the figure, and the vector itself is directed along the normal $\mathbf{n}$ to the area. Therefore

$$\mathbf{F}_1 ds\cdot\delta\mathbf{r} = d\mathbf{S}\cdot\boldsymbol{\Phi} = \Phi_n dS$$

i.e. the elementary work done by the force is equal to the flow of the vector Φ through the area element dS. Introduce a vector $\mathbf{A}$, which is determined by its vorticity and divergence in the whole infinite space

$$\operatorname{rot} \mathbf{A} = \mathbf{\Phi}, \qquad \operatorname{div} \mathbf{A} = 0 \tag{3.3}$$

(in the particular case of the motion of an electron it differs from the vector potential of the magnetic field only by a scalar multiplier). The sum of the elementary works done by the forces $\mathbf{F}_1 ds$ along virtual displacements is

$$\Sigma \mathbf{F}_1 ds \cdot \delta \mathbf{r} = \iint_B \Phi_n dS = \iint_B \operatorname{rot}_n \mathbf{A}\, dS$$

where the region B is bounded by the curves MPN and MQN. Applying Stokes' theorem we obtain

$$\Sigma \mathbf{F}_1 ds \cdot \delta \mathbf{r} = \int_{MPN} \mathbf{A} \cdot \boldsymbol{\tau} ds - \int_{MQN} \mathbf{A} \cdot \boldsymbol{\tau} ds = -\delta \int_{MPN} \mathbf{A} \cdot \boldsymbol{\tau} ds$$

The sum of the elementary works done by the forces P_1 and P_2 is

$$P_1 h_1 + P_2 h_2 = \mu \int_{MPN} ds - \mu \int_{MQN} ds = -\delta \int_{MPN} \mu\, ds$$

According to the principle of virtual displacements, the algebraic sum of the elementary works done by the above forces must be equal to zero, i.e.

$$\delta \int_{MPN} (\mathbf{A} \cdot \boldsymbol{\tau} + \mu)\, ds = 0 \tag{3.4}$$

Thus, the curve MPN, along which the integral

$$\int_M (\mathbf{A} \cdot \boldsymbol{\tau} + \mu)\, ds$$

assumes a stationary value, turns out to be:

1) The equilibrium figure of a string under the action of the force

$$\mathbf{F}_1 ds = -(\boldsymbol{\tau} \times \mathbf{\Phi})\, ds, \qquad \mathbf{\Phi} = \operatorname{rot} \mathbf{A}$$

At the points M and N the string is hanging over infinitely small ideal pulleys at the free ends of which the loads $P_1 = P_2 = \mu$ are attached.

2) The trajectory of a particle under the action of a gyroscopic force $\mathbf{F} = \mathbf{v} \times \mathbf{\Phi}$. The quantity of motion of the particle $\mu = mv$ is constant by virtue of Section 1 under the assumption that the mass depends only on

the velocity of the particle.

Consider as an illustrative example the problem of a focussing magnetic field. Let the intensity **H** of the field be directed parallel to the z-axis, and let H be a function of a single coordinate y. It is required to determine this function from the condition that the trajectories of the electrons, issuing from the given point $M(x_1, 0)$ of the Oxy-plane, pass through the given point $N(x_2, 0)$ of the same plane. Here $\mu = mv$ is a given quantity.

The corresponding mechanical problem is the following. Consider a cylindrical membrane [cloth], the lateral surface of which is closed and the free ends of which are hanging over horizontal rollers with axes parallel to the elements of the latter. The radii of the rollers are infinitely small, and at the ends of the membrane there are attached equal loads $P_1 = P_2 = \mu$. Find the law according to which the density of a heavy nonhomogeneous fluid filling up this vessel must change in order that there exists an infinity of equilibrium forms for the membrane.

4. Let us return to the trajectories of the particle subject to the action of a potential force $\mathbf{F}_0$ and a gyroscopic force $\mathbf{F}$. Without having recourse to the analogy with the equilibrium form of the string, let us show now that the trajectory of the particle, subject to the action of these forces, differs from all other curves passing through two given points by the fact that it assigns a stationary value to the action

$$S = \int_{t_1}^{t_2} L dt \qquad (L = \lambda + \mathbf{A}\cdot\mathbf{v} - V_0) \tag{4.1}$$

where L is the Lagrange function. The initial and final instants (t_1 and t_2) are given, and at these instants the moving particle must be at the given points. In relativistic mechanics we have, by virtue of (1.10)

$$L = -m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} + \mathbf{A}\cdot\mathbf{v} - V_0 \tag{4.2}$$

In the particular case of constant mass and a gyroscopic force equal to zero we have $A = 0$, $\lambda = (1/2)mv^2 = T$, i.e. we arrive at the principle of least action in the form of Hamilton and Ostrogradskii.

For the proof construct the Euler equations for the variational problem

$$\delta \int_{t_1}^{t_2} L dt = 0 \tag{4.3}$$

We have

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{d}{dt}\left(\frac{\partial \lambda}{\partial \dot{x}} + A_x\right) = \frac{d}{dt}\left(\frac{d\lambda}{dv}\cdot\frac{\partial v}{\partial \dot{x}} + A_x\right) = \frac{d}{dt}\left(mv\,\frac{\dot{x}}{v} + A_x\right) =$$

$$= \frac{d}{dt}(m\dot{x} + A_x) = -\frac{\partial V_0}{\partial x} + \frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_y}{\partial x}\dot{y} + \frac{\partial A_z}{\partial x}\dot{z}$$

$$\frac{d}{dt}(m\dot{x}) + \frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z} = -\frac{\partial V_0}{\partial x} + \frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_y}{\partial x}\dot{y} + \frac{\partial A_z}{\partial x}\dot{z}$$

Thus

$$\frac{d}{dt}(m\dot{x}) = -\frac{\partial V_0}{\partial x} + \dot{y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - \dot{z}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) = -\frac{\partial V_0}{\partial x} + v_y\Phi_z - v_z\Phi_y$$

and consequently

$$\frac{d}{dt}(m\mathbf{v}) = -\operatorname{grad} V_0 + \mathbf{v}\times\mathbf{\Phi}$$

The law of conservation (1.9) gives us

$$\lambda = 2T + V_0 - h, \qquad L = \lambda + \mathbf{A}\cdot\mathbf{v} - V_0 = 2T + \mathbf{A}\cdot\mathbf{v} - h$$

Thus, considering the mass as a function of the velocity only, time can be excluded. We then obtain

$$S = \int_{t_1}^{t_2} L\,dt = W - h(t_2 - t_1)$$

where

$$W = \int_{t_1}^{t_2} (mv^2 + \mathbf{A}\cdot\mathbf{v})\,dt, \qquad \text{or} \qquad W = \int_{M}^{N} (mv + A\cdot\boldsymbol{\tau})\,dt \tag{4.4}$$

Thus, the trajectory of a particle moving under the action of a potential and a gyroscopic force assigns a stationary value to the integral (4.4) in comparison with all other curves, passing through the given points M and N. Here we have set

$$mv\,ds = mv\sqrt{x^2 + y^2 + z^2}\,ds$$

and the velocity v must be found from the law of conservation (1.9) as a function of the coordinates for a given h. If a gyroscopic force only is acting, then $q = mv = \text{const}$ and we arrive again at the equality (3.4).

In relativistic mechanics we have, by virtue of (1.11)

$$h = \frac{m_0c^2}{\sqrt{1 - v^2/c^2}} + V_0$$

From here it follows that

$$v = c\sqrt{1 - \left(\frac{m_0 c^2}{h - V_0}\right)^2}, \qquad mv = \frac{1}{c}\sqrt{(h - V_0)^2 - m_0{}^2 c^4}$$

and thus

$$W = \int_M^N \left\{\frac{1}{c}\sqrt{(h - V_0)^2 - m_0{}^2 c^4} + \mathbf{A}\cdot\boldsymbol{\tau}\right\} dt \tag{4.5}$$

If we set $A = 0$, $h = m_0 c^2 + h_1$ and let $c \to \infty$, then we arrive at the principle of least action in the form of Maupertuis and Euler.

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